

15. (a) Let $f(z)$ is analytic and one-to-one on an open set D . Prove that the inverse map $g(w) = f^{-1}(w)$ is analytic on $f(D)$.

Or

- (b) Prove that if $f(z)$ is analytic on a convex domain D such that $\operatorname{Re} f'(z) > 0$ in D , then $f(z)$ is univalent in D .

SECTION C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

16. State and prove arguments principles.
17. Show that $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2 + 1} dx = \frac{\pi(1 - e^{-2})}{2}$.
18. State and prove Schwarz's reflection principles for analytic function.
19. For $\operatorname{Re} \zeta > 1$ Prove that $\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz$.
20. State and prove schottky's theorem.

APRIL/MAY 2025

**GMA41/DMA41 — COMPLEX
ANALYSIS - II**

Time : Three hours

Maximum : 75 marks

SECTION A — ($10 \times 2 = 20$ marks)

Answer ALL questions.

1. Find the residue of $f(z) = \frac{1}{(z^3 - 1)(z + 1)^2}$ at $z = -1$.
2. State Cauchy's residue theorem.
3. Find the Cauchy's principal value $\int_{-\infty}^{\infty} x dx$.
4. Evaluate $\int_0^{2\pi} \sin^{2n} \theta d\theta$.
5. Define analytic continuation.



6. Define natural boundary.
7. State Mittag-Leffler theorem.
8. Show that $\Gamma(x+1) = x\Gamma(x)$.
9. State local mapping theorem.
10. Show that $f(z) = \frac{z}{(1-z)^3}$ is univalent at $\frac{1}{2}$.

SECTION B — (5 × 5 = 25 marks)

Answer ALL questions.

11. (a) Evaluate $\int_{|z|=r} \frac{\sin(z^2)}{z^2(z-a)} dz$.

Or

- (b) Evaluate $I = \int_{|z|=R} f(z) dz$ where

$$f(z) = \frac{z^{2n} + 3m - 1}{(z^2 + a)^n (z^3 + b)^m} \quad \text{where}$$

$a, b \in C \setminus \{0\}$, $R > \max \{ \sqrt[n]{|a|}, |b|^{1/3} \}$ and m and n are fixed positive integers.

12. (a) Evaluate $I = \int_0^{2\pi} \frac{d\theta}{1 + \alpha^2 - 2\alpha \cos \theta}$, $1 \neq \alpha > 0$.

Or

- (b) Let $\gamma_\epsilon = \{z : |z - z_0| = \epsilon \text{ and } \theta_1 \leq \arg(z - z_0) \leq \theta_2\}$. If f is continuous on $0 < |z - z_0| \leq \epsilon$ and if $\lim_{z \rightarrow z_0} (z - z_0) f(z) = l$, prove that

$$\lim_{\epsilon \rightarrow 0} \int_{\gamma_\epsilon} f(z) dz = il(\theta_2 - \theta_1) \quad \text{where } \gamma_\epsilon \text{ is positively oriented.}$$

- (a) State and prove theorem on solution to a Dirichlet problem.

Or

- (b) Let $u : D \rightarrow R$ be continuous on a domain D such that for each point

$$a \in D, u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta \text{ holds whenever}$$

$\bar{\Delta}(a; r) \subset D$. Prove that $u(z)$ is harmonic on D .

14. (a) Prove that for $p \in N \cup \{0\}$, we have

$$|1 - E_p(z)| \leq |z|^{p+1} \text{ for all } |z| \leq 1.$$

Or

- (b) State and prove Weierstrass Factorization theorem.